NONSTANDARD SOLUTIONS OF THE YANG-BAXTER EQUATION

ANTHONY GIAQUINTO AND TIMOTHY J. HODGES

Abstract. Explicit solutions of the quantum Yang-Baxter equation are given corresponding to the non-unitary solutions of the classical Yang-Baxter equation for $\mathfrak{sl}(5)$.

1. Introduction

Etingof and Kazhdan recently proved that any finite dimensional Lie bialgebra $\mathfrak{g}$ may be quantized [3]. That is, there exists a topological Hopf algebra structure on $U(\mathfrak{g})[[h]]$ such that the Lie bialgebra structure on $\mathfrak{g}$ is the one induced on $\mathfrak{g}$ by passing to the "semi-classical limit". From this they deduced a general procedure for quantizing solutions of the classical Yang-Baxter equation (CYBE). Thus, at least in theory, one can construct solutions of the quantum Yang-Baxter equation from given solutions of the classical Yang-Baxter equation. Unfortunately, their procedure is not easy to implement explicitly, even in small dimensional situations.

In this note we exhibit an explicit answer to this problem for a particularly interesting family of Lie bialgebra structures on $\mathfrak{sl}(5)$. These are the bialgebra structures associated to non-unitary solutions of the CYBE (or equivalently of the modified classical Yang-Baxter equation (MCYBE)) as classified by Belavin and Drinfeld [1]. For each such solution of the CYBE we construct an R-matrix using the Gerstenhaber-Giaquinto-Schack (GGS) conjecture [4]. The YBE was verified in each case using *Mathematica*.

The GGS conjecture concerns the form of the quantization of such solutions of the CYBE in the case of $\mathfrak{sl}(n)$. The case of $\mathfrak{sl}(5)$ is to some extent the first interesting case. For $\mathfrak{sl}(2)$ there are no solutions of the MCYBE except the standard one. For $\mathfrak{sl}(3)$ the only non-standard solution is that associated to the well-known Cremmer-Gervais quantization and for $\mathfrak{sl}(4)$ the nonstandard solutions are essentially of three types, the Cremmer-Gervais solution and two other fairly simple examples. The corresponding R-matrices for the latter two types can be constructed using other techniques [6]. On the other hand for $\mathfrak{sl}(5)$ there are 13 different types of solutions to the MCYBE and for many of these the corresponding R-matrix was hitherto unknown. The validity of the GGS conjecture for $\mathfrak{sl}(5)$ gives strong evidence that the conjecture should be true for all n.

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2. Solutions to the CYBE and quantization

2.1. **The Belavin-Drinfeld description of solutions to the CYBE.** Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra. Let Δ be the associated root system and Γ a set of simple roots. A classical r-matrix over $\mathfrak{g}$ is a an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfying the classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

Take an invariant bilinear from on form on $\mathfrak{g}$ and let $t \in \mathfrak{g} \otimes \mathfrak{g}$ be the associated Casimir element. In [1] Belavin and Drinfeld gave the following description of solutions of the CYBE which satisfy $r_{12} + r_{21} = t$. These are the "non-unitary" solutions.

Let Γ_1, Γ_2 be two subsets of Γ and let $\tau : \Gamma_1 \to \Gamma_2$ be a bijection satisfying

1. $(\tau\alpha, \tau\beta) = (\alpha, \beta)$ for all $\alpha, \beta \in \Gamma$;
2. For every $\alpha \in \Gamma_1$, there is a $k \geq 0$ with $\tau^k\alpha \in \Gamma_1$ but $\tau^{k+1}\alpha \notin \Gamma_1$.

The data $(\tau, \Gamma_1, \Gamma_2)$ (or more concisely just τ) is often called a *Belavin-Drinfeld triple*. Given such a triple τ, an element $r^0 \in \mathfrak{h} \otimes \mathfrak{h}$ is called τ-admissible if

1. $r^0_{12} + r^0_{21} = t^0$
2. $(\tau\alpha \otimes 1)r^0 + (1 \otimes \alpha)r^0 = t^0$

where t^0 is the component of t in $\mathfrak{h} \otimes \mathfrak{h}$. A τ-admissible r^0 is necessarily of the form $t^0/2 + \tilde{r}^0$ where $\tilde{r}^0 \in \mathfrak{h} \wedge \mathfrak{h}$. The set of all $\tilde{r}^0$ forms a linear subvariety of $\mathfrak{h} \wedge \mathfrak{h}$ of dimension $\binom{d}{2}$ where $d = \#(\Gamma - \Gamma_1)$.

Now τ can be extended to an isomorphism of Lie subalgebras $\tau : \mathfrak{g}_1 \to \mathfrak{g}_2$ where $\mathfrak{g}_i$ is the Lie subalgebra of $\mathfrak{g}$ associated to Γ_i. Choose $e_\alpha \in \mathfrak{g}_\alpha$ such that $(e_\alpha, e_{-\alpha}) = 1$ and $\tau(e_\alpha) = e_{\tau\alpha}$ and define an ordering on Δ by $\alpha \prec \beta$ if $\tau^k\alpha = \beta$ for some positive integer k. View $\mathfrak{g} \wedge \mathfrak{g}$ as a subset of $\mathfrak{g} \otimes \mathfrak{g}$ via the identification $x \wedge y = 1/2(x \otimes y - y \otimes x)$. Then Belavin and Drinfeld showed [1] that

$$r = r^0 + \sum_{\alpha > 0} e_{-\alpha} \otimes e_\alpha + \sum_{\substack{\alpha,\beta > 0 \\ \alpha \prec \beta}} e_{-\alpha} \wedge e_\beta$$

is a solution of the Yang-Baxter equation satisfying $r_{12} + r_{21} = t$ and that every such solution is of this form for some choice of $\mathfrak{h}$, Γ, τ and r^0.

For any $\mathfrak{g}$ there is the "trivial" triple which has $\Gamma_1 = \Gamma_2 = \emptyset$ and $\tilde{r}^0 \in \mathfrak{h} \wedge \mathfrak{h}$ arbitrary. A particularly interesting triple for $\mathfrak{sl}(n)$ is the "Cremmer-Gervais" triple which has $\Gamma_1 = \{\alpha_2, \alpha_3, \ldots, \alpha_{n-1}\}$, $\quad \Gamma_2 = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-2}\}$, and $\tau(\alpha_i) = \alpha_{i-1}$. In contrast to the trivial triple, there is a unique admissible r^0 for the Cremmer-Gervais triple.

2.2. **The Gerstenhaber-Giaquinto-Schack conjecture.** The Gerstenhaber-Giaquinto-Schack conjecture is a conjectured form for the quantization of the above classical r-matrices in the case where $\mathfrak{g} = \mathfrak{sl}(n)$, considered as a subset of $M_n(\mathbb{C})$. In this setting, a quantization of a classical r-matrix is an $R \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ which has semi-classical limit r and satisfies the quantum Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.

Take the form to be the trace form $(x, y) = \mathrm{Tr}(xy)$ and let $\mathfrak{h}$ be the Cartan subalgebra consisting of diagonal matrices of trace zero. The standard Cartan-Weyl basis is then $e_{\alpha_i} = e_{i,i+1}$, $\quad e_{-\alpha_i} = e_{i+1,i}$ and $h_{\alpha_i} = [e_{\alpha_i}, e_{-\alpha_i}] = e_{ii} - e_{i+1,i+1}$. Let

τ be a Belavin-Drinfeld triple as described above and let $r^0 \in \mathfrak{h}\otimes\mathfrak{h}$ be τ-admissible. Set

$$a = \sum_{\substack{\alpha,\beta>0 \\ \alpha\prec\beta}} e_{-\alpha} \wedge e_\beta$$

and

$$c_+ = \sum_{\alpha>0} e_{-\alpha} \otimes e_\alpha, \quad c = \sum_{\alpha>0} e_{-\alpha} \wedge e_\alpha.$$

Set $\epsilon = -(ac + ca + a^2)$. Now define $\tilde{a}$ by

$$\tilde{a} = \sum a_{jl}^{ik} q^{a_{jl}^{ik} \epsilon_{jl}^{ik}} e_{ij} \otimes e_{kl}$$

where $a = \sum a_{jl}^{ik} e_{ij} \otimes e_{kl}$ and similarly for ϵ. Set $\hat{q} = q - q^{-1}$. The standard R-matrix is then

$$R_s = q^{t^0+1/n} + \hat{q}c_+ = q\sum_i e_{ii}\otimes e_{ii} + \sum_{i\neq j} e_{ii}\otimes e_{jj} + \hat{q}\sum_{i>j} e_{ij}\otimes e_{ji}.$$

It is easy to check that R_s satisfies the quantum Yang-Baxter equation and that PR_s satisfies the Hecke relation $(PR_s - q)(PR_s + q^{-1}) = 0$ where P is the permutation matrix.

Gerstenhaber-Giaquinto-Schack Conjecture. *Let τ be a Belavin-Drinfeld triple for $\mathfrak{sl}(n)$ and suppose $r^0 = t^0/2 + \tilde{r}^0$ is τ-admissible. Then the matrix*

$$R = q^{\tilde{r}^0}(R_s + \hat{q}\,\tilde{a})\,q^{\tilde{r}^0}$$

satisfies the quantum Yang-Baxter equation and PR satisfies the Hecke relation.

Taking τ to be the trivial triple yields the standard R-matrix when $r^0 = t^0/2$ and the standard multiparameter R-matrices when r^0 is arbitrary. For use later, let $R(r^0) = q^{\tilde{r}^0}(R_s)q^{\tilde{r}^0}$ denote the standard multiparameter R-matrix. As is well known, if $r^0 = t^0/2 + \sum_{i<j} c_{ij}\, e_{ii}\wedge e_{jj}$ then

$$R(r^0) = q\sum_i e_{ii}\otimes e_{ii} + \sum_{i<j}(q^{c_{ij}} e_{ii}\otimes e_{jj} + q^{-c_{ij}} e_{jj}\otimes e_{ii}) + \hat{q}\sum_{i>j} e_{ij}\otimes e_{ji}.$$

For the Cremmer-Gervais triples described above the formula gives the Cremmer-Gervais R-matrices [2].

2.3. **The GGS conjecture for $\mathfrak{sl}(5)$.** We now consider the explicit form of the R-matrices associated to the Belavin-Drinfeld triples on $\mathfrak{sl}(5)$. According to the GGS Conjecture, each R is of the form $R(r^0) + \hat{q}\, q^{\tilde{r}^0}\tilde{a}q^{\tilde{r}^0}$ for an admissible r^0. The specific form of $R(r^0)$ has already been exhibited. The other summand, $q^{\tilde{r}^0}\tilde{a}q^{\tilde{r}^0}$, is always a sum of "quantized" wedge products. Specifically, for positive roots α and β and any constant c, set $e_{-\alpha}\wedge_c e_\beta = q^{-c}e_{-\alpha}\otimes e_\beta - q^c e_\beta\otimes e_{-\alpha}$. For all triples, the term $q^{\tilde{r}^0}\tilde{a}q^{\tilde{r}^0}$ is always of the form $\sum_{\substack{\alpha,\beta>0 \\ \alpha\prec\beta}} e_{-\alpha}\wedge_{c(\alpha,\beta)} e_\beta$ where the constants $c(\alpha,\beta)$ are determined by $\tilde{r}^0$ and ϵ.

Denote by $\mathcal{T}$ the set of triples on $\mathfrak{sl}(5)$. Notice that if (τ,Γ_1,Γ_2) is a triple, then $(\tau^{-1},\Gamma_2,\Gamma_1)$ is also a triple. Also the graph automorphism of A_4 induces a bijection on the set of triples. Since these two involutions of $\mathcal{T}$ commute, this gives an action of the group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{T}$.

Proposition 2.1. *The Gerstenhaber-Giaquinto-Schack conjecture is true for $n = 5$. The triples below comprise a complete set of representatives from the 13 orbits under the action of $C_2 \times C_2$ on $\mathcal{T}$. For each triple the generic admissible r^0 and the Hecke R-matrix produced by the GGS conjecture are also explicitly given.*

1. $|\Gamma_1| = 3$

   (a) *The "Cremmer-Gervais" triple:* $\Gamma_1 = \{\alpha_2, \alpha_3, \alpha_4\}$, $\quad \Gamma_2 = \{\alpha_1, \alpha_2, \alpha_3\}$, $\tau(\alpha_i) = \alpha_{i-1}$.

$$\begin{aligned} r^0 = t^0/2 + \frac{1}{5}(&-3\, h_{\alpha_1} \wedge h_{\alpha_2} - 4\, h_{\alpha_1} \wedge h_{\alpha_3} - 3\, h_{\alpha_1} \wedge h_{\alpha_4} \\ &- 4\, h_{\alpha_2} \wedge h_{\alpha_3} - 4\, h_{\alpha_2} \wedge h_{\alpha_4} - 3\, h_{\alpha_3} \wedge h_{\alpha_4}) \end{aligned}$$

$$\begin{aligned} R = R(r^0) + \hat{q}\, (&e_{54} \wedge_{2/5} e_{34} + e_{54} \wedge_{4/5} e_{23} + e_{54} \wedge_{6/5} e_{12} + e_{43} \wedge_{2/5} e_{23} \\ &+ e_{43} \wedge_{4/5} e_{12} + e_{32} \wedge_{2/5} e_{12} + e_{53} \wedge_{2/5} e_{24} + e_{53} \wedge_{4/5} e_{13} \\ &+ e_{42} \wedge_{2/5} e_{13} + e_{52} \wedge_{2/5} e_{14}) \end{aligned}$$

   (b) *The "generalized Cremmer-Gervais" triple:* $\Gamma_1 = \{\alpha_1, \alpha_3, \alpha_4\}$, $\Gamma_2 = \{\alpha_1, \alpha_2, \alpha_4\}$, $\quad \tau(\alpha_i) = \alpha_j$, *where* $j \equiv i + 3$ *(mod 5).*

$$\begin{aligned} r^0 = t^0/2 + \frac{1}{5}(&h_{\alpha_1} \wedge h_{\alpha_2} - 2\, h_{\alpha_1} \wedge h_{\alpha_3} + h_{\alpha_1} \wedge h_{\alpha_4} \\ &- 2\, h_{\alpha_2} \wedge h_{\alpha_3} - 2\, h_{\alpha_2} \wedge h_{\alpha_4} + h_{\alpha_3} \wedge h_{\alpha_4}) \end{aligned}$$

$$\begin{aligned} R = R(r^0) + \hat{q}\, (&e_{54} \wedge_{2/5} e_{23} + e_{21} \wedge_{4/5} e_{23} + e_{43} \wedge_{6/5} e_{23} + e_{21} \wedge_{2/5} e_{45} \\ &+ e_{43} \wedge_{4/5} e_{45} + e_{43} \wedge_{2/5} e_{12} + e_{53} \wedge_{2/5} e_{13}) \end{aligned}$$

2. $|\Gamma_1| = 2$

   (a) $\Gamma_1 = \{\alpha_3, \alpha_4\}$, $\quad \Gamma_2 = \{\alpha_1, \alpha_2\}$, $\quad \tau(\alpha_i) = \alpha_{i-2}$

$$\begin{aligned} r^0 = t^0/2 &+ c\, h_{\alpha_1} \wedge h_{\alpha_2} + ((c-1)/2)\, h_{\alpha_1} \wedge h_{\alpha_3} + c\, h_{\alpha_1} \wedge h_{\alpha_4} \\ &- ((1+3c)/4)\, h_{\alpha_2} \wedge h_{\alpha_3} + ((c-1)/2)\, h_{\alpha_2} \wedge h_{\alpha_4} + c\, h_{\alpha_3} \wedge h_{\alpha_4} \end{aligned}$$

$$R = R(r^0) + \hat{q}\, (e_{43} \wedge_{(3+c)/8} e_{12} + e_{54} \wedge_{(3+c)/8} e_{23} + e_{53} \wedge_{(1-c)/2} e_{13})$$

   (b) $\Gamma_1 = \{\alpha_3, \alpha_4\}$, $\quad \Gamma_2 = \{\alpha_1, \alpha_2\}$, $\quad \tau(\alpha_i) = \alpha_{5-i}$

$$\begin{aligned} r^0 = t^0/2 + \frac{1}{5}(&-c\, h_{\alpha_1} \wedge h_{\alpha_2} - (1+c)\, h_{\alpha_1} \wedge h_{\alpha_3} - 3\, h_{\alpha_1} \wedge h_{\alpha_4} \\ &- 2\, h_{\alpha_2} \wedge h_{\alpha_3} + (c-1)\, h_{\alpha_2} \wedge h_{\alpha_4} + c\, h_{\alpha_3} \wedge h_{\alpha_4}) \end{aligned}$$

$$R = R(r^0) + \hat{q}\,(e_{54} \wedge_{3/5} e_{12} + e_{43} \wedge_{4/5} e_{23} + e_{53} \wedge_{-9/5} (-e_{13}))$$

(c) $\Gamma_1 = \{\alpha_2, \alpha_4\}, \quad \Gamma_2 = \{\alpha_1, \alpha_3\}, \quad \tau(\alpha_i) = \alpha_{i-1}$

$$\begin{aligned} r^0 &= t^0/2 + c\, h_{\alpha_1} \wedge h_{\alpha_2} + (1+3c)\, h_{\alpha_1} \wedge h_{\alpha_3} + (8c/3+1)\, h_{\alpha_1} \wedge h_{\alpha_4} \\ &\quad (1+3c)\, h_{\alpha_2} \wedge h_{\alpha_3} + (1+3c)\, h_{\alpha_2} \wedge h_{\alpha_4} + c\, h_{\alpha_3} \wedge h_{\alpha_4} \end{aligned}$$

$$R = R(r^0) + \hat{q}\,(e_{32} \wedge_{1+c} e_{12} + e_{54} \wedge_{1+c} e_{34})$$

(d) $\Gamma_1 = \{\alpha_2, \alpha_4\}, \quad \Gamma_2 = \{\alpha_1, \alpha_3\}, \quad \tau(\alpha_4) = \alpha_1, \quad \tau(\alpha_2) = \alpha_3$

$$\begin{aligned} r^0 &= t^0/2 + \frac{1}{5}((2-c)\, h_{\alpha_1} \wedge h_{\alpha_2} + (1-c)\, h_{\alpha_1} \wedge h_{\alpha_3} - 3\, h_{\alpha_1} \wedge h_{\alpha_4} \\ &\quad + 2\, h_{\alpha_2} \wedge h_{\alpha_3} + (c-1)\, h_{\alpha_2} \wedge h_{\alpha_4} + c\, h_{\alpha_3} \wedge h_{\alpha_4}) \end{aligned}$$

$$R = R(r^0) + \hat{q}\,(e_{54} \wedge_{2/5} e_{12} + e_{32} \wedge_{2/5} e_{34})$$

(e) $\Gamma_1 = \{\alpha_1, \alpha_3\}, \quad \Gamma_2 = \{\alpha_1, \alpha_4\}, \quad \tau(\alpha_i) = \alpha_j$, *where* $j \equiv i + 3$ *(mod 5).*

$$\begin{aligned} r^0 &= t^0/2 + ((1-3c)/2)\, h_{\alpha_1} \wedge h_{\alpha_2} + ((c-1)/2)\, h_{\alpha_1} \wedge h_{\alpha_3} + c\, h_{\alpha_1} \wedge h_{\alpha_4} \\ &\quad + (3c-1)\, h_{\alpha_2} \wedge h_{\alpha_3} + (3c-1)\, h_{\alpha_2} \wedge h_{\alpha_4} + c\, h_{\alpha_3} \wedge h_{\alpha_4} \end{aligned}$$

$$R = R(r^0) + \hat{q}\,(e_{43} \wedge_{(1+3c)/4} e_{12} + e_{21} \wedge_{(1+3c)/4} e_{45} + e_{43} \wedge_{1-c} e_{45})$$

(f) $\Gamma_1 = \{\alpha_3, \alpha_4\}, \quad \Gamma_2 = \{\alpha_1, \alpha_2\}, \quad \tau(\alpha_i) = \alpha_{i-1}$

$$\begin{aligned} r^0 &= t^0/2 + ((c-3)/6)\, h_{\alpha_1} \wedge h_{\alpha_2} + ((c-1)/2)\, h_{\alpha_1} \wedge h_{\alpha_3} + c\, h_{\alpha_1} \wedge h_{\alpha_4} \\ &\quad + ((c-1)/2)\, h_{\alpha_2} \wedge h_{\alpha_3} + (4c/3)\, h_{\alpha_2} \wedge h_{\alpha_4} + c\, h_{\alpha_3} \wedge h_{\alpha_4} \end{aligned}$$

$$R = R(r^0) + \hat{q}\,(e_{32} \wedge_{1/2+c/6} e_{12} + e_{43} \wedge_{1/2+c/6} e_{23} + e_{43} \wedge_{1+c/3} e_{12})$$

3. $|\Gamma_1| = 1$

(a) $\Gamma_1 = \{\alpha_1\}, \quad \Gamma_2 = \{\alpha_2\}, \quad \tau(\alpha_1) = \alpha_2$

$$\begin{aligned} r^0 &= t^0/2 + ((1+y)/3)\, h_{\alpha_1} \wedge h_{\alpha_2} + y\, h_{\alpha_1} \wedge h_{\alpha_3} + ((3z-x)/3)\, h_{\alpha_1} \wedge h_{\alpha_4} \\ &\quad + y\, h_{\alpha_2} \wedge h_{\alpha_3} + z\, h_{\alpha_2} \wedge h_{\alpha_4} + x\, h_{\alpha_3} \wedge h_{\alpha_4} \end{aligned}$$

$$R = R(r^0) + \hat{q}\,(e_{21} \wedge_{(2-y)/3} e_{23})$$

(b) $\Gamma_1 = \{\alpha_1\}, \quad \Gamma_2 = \{\alpha_3\}, \quad \tau(\alpha_1) = \alpha_3$

$$\begin{aligned} r^0 = {} & t^0/2 + ((z-2y)/2)\, h_{\alpha_1} \wedge h_{\alpha_2} + ((1+x)/2)\, h_{\alpha_1} \wedge h_{\alpha_3} + x\, h_{\alpha_1} \wedge h_{\alpha_4} \\ & + y\, h_{\alpha_2} \wedge h_{\alpha_3} + z\, h_{\alpha_2} \wedge h_{\alpha_4} + x\, h_{\alpha_3} \wedge h_{\alpha_4} \end{aligned}$$

$$R = R(r^0) + \hat{q}\,(e_{21} \wedge_{(z-2)/4} e_{34})$$

(c) $\Gamma_1 = \{\alpha_1\}, \quad \Gamma_2 = \{\alpha_4\}, \quad \tau(\alpha_1) = \alpha_4$

$$\begin{aligned} r^0 = {} & t^0/2 + ((y-2z)/2)\, h_{\alpha_1} \wedge h_{\alpha_2} + ((y-2x)/2) h_{\alpha_1} \wedge h_{\alpha_3} + ((1-x+z)/2)\, h_{\alpha_1} \wedge h_{\alpha_4} \\ & + y\, h_{\alpha_2} \wedge h_{\alpha_3} + z\, h_{\alpha_2} \wedge h_{\alpha_4} + x\, h_{\alpha_3} \wedge h_{\alpha_4} \end{aligned}$$

$$R = R(r^0) + \hat{q}\,(e_{21} \wedge_{(y+2)/4} e_{45})$$

(d) $\Gamma_1 = \{\alpha_2\}, \quad \Gamma_2 = \{\alpha_3\}, \quad \tau(\alpha_2) = \alpha_3$

$$\begin{aligned} r^0 = {} & t^0/2 + (-1+3y-z)\, h_{\alpha_1} \wedge h_{\alpha_2} + (-1-x+3y) h_{\alpha_1} \wedge h_{\alpha_3} + 3(z-x)\, h_{\alpha_1} \wedge h_{\alpha_4} \\ & + y\, h_{\alpha_2} \wedge h_{\alpha_3} + z\, h_{\alpha_2} \wedge h_{\alpha_4} + x\, h_{\alpha_3} \wedge h_{\alpha_4} \end{aligned}$$

$$R = R(r^0) + \hat{q}\,(e_{32} \wedge_{1-x-y+z} e_{34})$$

4. $|\Gamma_1| = 0$ *The "trivial triple:"* $\Gamma_1 = \Gamma_2 = \emptyset$

$r^0 = t^0/2 + \tilde{r}^0$ *with* $\tilde{r}^0 \in \mathfrak{h} \wedge \mathfrak{h}$ *arbitrary.*

$R = R(r^0)$ *is the standard multiparameter* R*-matrix.*

Perhaps the most interesting new R-matrix is that associated to type $1\,(b)$, the generalized Cremmer-Gervais triple. Like the Cremmer-Gervais triple, its Γ_1, which must omit at least one root, omits precisely one and thus its r^0 is uniquely determined. Setting $\hat{p} = -\hat{q}$, the matrix form of the generalized Cremmer-Gervais R-matrix is

$$
\begin{pmatrix}
q&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&q^{\frac15}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&q^{-\frac35}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&q^{\frac35}&0&0&0&\hat{p}q^{\frac25}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&q^{-\frac15}&0&0&0&0&0&0&0&\hat{p}q^{\frac25}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&\hat{q}&0&0&0&q^{-\frac15}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\hat{q}q^{-\frac45}&0&0&0&q&0&0&0&\hat{p}q^{\frac45}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&q^{\frac15}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\hat{q}q^{-\frac25}&0&0&0&q^{-\frac35}&0&0&0&\hat{p}q^{\frac65}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&q^{\frac35}&0&0&0&\hat{p}q^{\frac25}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\hat{q}&0&0&0&0&0&0&0&q^{\frac35}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\hat{q}&0&0&0&q^{-\frac15}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&q&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&q^{\frac15}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&q^{-\frac35}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\hat{q}&0&0&0&0&0&0&0&\hat{q}q^{-\frac25}&0&0&0&q^{-\frac35}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\hat{q}&0&0&0&\hat{q}q^{-\frac65}&0&0&0&q^{\frac35}&0&0&0&\hat{p}q^{\frac25}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\hat{q}&0&0&0&q^{-\frac15}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\hat{q}q^{-\frac45}&0&0&0&q&0&0&0&\hat{p}q^{\frac45}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&q^{\frac15}&0&0&0&0&0\\
0&0&0&0&\hat{q}&0&0&0&0&0&0&0&\hat{q}q^{-\frac25}&0&0&0&0&0&0&0&q^{\frac15}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\hat{q}&0&0&0&0&0&0&0&\hat{q}q^{-\frac25}&0&0&0&q^{-\frac35}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\hat{q}&0&0&0&0&0&0&0&q^{\frac35}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\hat{q}&0&0&0&q^{-\frac15}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&q
\end{pmatrix}.
$$

3. Conclusion

We have constructed here quantizations of each type of non-unitary solution of the classical Yang-Baxter equation for $\mathfrak{sl}(5)$. In so doing we verified in this case the conjecture of Gerstenhaber, Giaquinto and Schack. This gives further evidence that the GGS conjecture should be true for all Belavin-Drinfeld triples on $\mathfrak{sl}(n)$.

One can proceed in the usual way to construct for each of these R, a quantization of $\mathbb{C}[SL(5)]$, the algebra of algebraic functions on $SL_{\mathbb{C}}(5)$. First one constructs the associated bialgebra $A(R)$. Using a case-by-case analysis one can see that the Poincare series of the associated quantum space and exterior algebra are the same as in the commutative case. Thus $A(R)$ contains a group-like q-determinant element D which turns out to be central. Hence one may define a Hopf algebra structure on $\mathbb{C}_R[SL(5)] = A(R)/(D-1)$. Since R is a Hecke symmetry in the sense of Gurevich, it is possible to exploit some Hecke algebra techniques to show that the category of comodules over these Hopf algebras is equivalent as a rigid monoidal category to the category of comodules over $\mathbb{C}_q[SL(5)]$ [5, 7, 8]. Hence these R-matrices do produce genuine nonstandard quantizations of $\mathbb{C}[SL(5)]$.

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Department of Mathematics, Texas A & M University, College Station, TX 77843-3368

E-mail address: tonyg@math.tamu.edu

University of Cincinnati, Cincinnati, OH 45221-0025, U.S.A.

E-mail address: timothy.hodges@uc.edu